ON THE PROBLEM OF OPTIMUM ROCKET TRAJECTORIES

(K ZADACHE OB OPTIMAL NYKH TRAEKTORIIAKH RAKETY)

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Extremum problems related to the change of the velocity vector $\mathbf{V}$ of an ideally guided rocket by a vector $\Delta\mathbf{V}$ in a constant gravitational field have been studied in [1 to 3].

In this paper the solution proposed by Gorelov [1] for the problem of the maximum powered flight duration $\Delta t = t_f - t_i$ in a horizontal plane for a prescribed final mass m_f is derived from the equations for the extremals obtained by Leitmann [2]. A study is made of the special case in which for certain trajectory phases the thrust is used only to maintain constant altitude.

For the case of flight in a vertical plane, it is shown that Gorelov's condition β/m = const , where $\beta = -m^{\cdot}$, will not be necessary in the general case. On the other hand, we wish to find $\max m_f$ for an unspecified t_f, for some class of boundary conditions we shall have $\beta \neq \beta_{max}$.

1. Flight in a horizontal plane. As is known [1 and 2], the solution of the problem of finding $\max t_f$ for a prescribed m_f coincides with the solution of the problem of finding $\max m_f$ for a specified t_f . In [2] it is noted that an intermediate thrust may appear in the solution of the latter problem. In this case β is determined by the fifth equation in (8) and by Equation (20). If we take Equation (23) into consideration, the above equations may be written as

$$\lambda_m = \frac{c^2\beta}{m^2}\left[\left(\frac{c\beta}{m}\right)^2 - g^2\right]^{-1/2}\left[\left(\frac{\lambda_\gamma}{V}\right)^2 + \lambda_V^2\right]^{1/2} \qquad (1.1)$$

The third equation in (8), taking (20) and (23) into consideration, may be written as

$$\lambda_m = \frac{c^2\beta^2}{m^3}\left[\left(\frac{c\beta}{m}\right)^2 - g^2\right]^{-1/2}\left[\left(\frac{\lambda_\gamma}{V}\right)^2 + \lambda_V^2\right]^{1/2} \qquad (1.2)$$

It is directly verifiable that

$$\left(\frac{\lambda_\gamma}{V}\right)^2 + \lambda_V^2 = \mathrm{const} \qquad (1.3)$$

Dividing (1.2) by (1.1), we find

$$\frac{\lambda_m^{\cdot}}{\lambda_m} = \frac{\beta}{m}, \qquad \text{or} \qquad \lambda_m = \frac{c_1}{m} \qquad (1.4)$$

Substituting the last expression into (1.1) and taking (1.3) into consideration, after some simple transformations we obtain

$$\frac{\beta}{m} = \text{const} \tag{1.5}$$

as in Gorelov's solution.

If the restriction $\beta \leqslant \beta_{max}$ is imposed, this regime will evidently have place in the entire trajectory, provided that the condition $\beta(t_i) \leqslant \beta_{max}$. is satisfied. We shall write this in explicit form.

Following Lur'e [3], we write the equations of motion in vector form

$$\mathbf{V}^{\cdot} = \left[\left(\frac{c\beta}{m}\right)^2 - g^2\right]^{1/2}\mathbf{e}, \qquad \mathbf{e}\cdot\mathbf{e} = 1, \qquad m^{\cdot} = -\beta \tag{1.6}$$

In [3] it is shown that $\mathbf{e}$ = const . If we also take (1.5) into account and integrate, we find

$$\frac{\beta}{m} = \frac{1}{\Delta t}\ln\frac{m_i}{m_f}, \qquad \Delta V = |\Delta\mathbf{V}| = \left[\left(c\ln\frac{m_i}{m_f}\right)^2 - (g\Delta t)^2\right]^{1/2} \tag{1.7}$$

The condition appears in the form

$$\beta(t_i) = m_i g \ln\frac{m_i}{m_f}\left[\left(c\ln\frac{m_i}{m_f}\right)^2 - (\Delta V)^2\right]^{-1/2} \leqslant \beta_{max} \tag{1.8}$$

To obtain a final solution of the problem of finding max t_f for a prescribed m_f, we must check the resulting solution for the optimum capacity in comparison with the solution containing arcs of power-off flight, where $\beta = mg/c$, since on these arcs Equations (1.1) and (1.2) become meaningless. It should be noted that for this case Formula (1.6) of [1] is also inapplicable.

Let the trajectory consist of two arcs such that on the first arc β/m const $\neq g/c$ and the mass varies from m_i to m_1 , while on the second arc $\beta = mg/c$ and the mass varies from m_1 to m_f . Then the total time of flight will correspondingly consist of two terms

$$\Delta t = \frac{1}{g}\left[\left(c\ln\frac{m_i}{m_1}\right)^2 - (\Delta V)^2\right]^{1/2} + \frac{c}{g}\ln\frac{m_1}{m_f} \tag{1.9}$$

Calculating the derivative $\partial\Delta t/\partial m_1$, we find that it is negative. If we interchange the positions of the arcs, the derivative becomes positive. Thus, any inclusion of an arc with $\beta = mg/c$ in the trajectory will reduce t_f .

2. Flight in a vertical plane. Using the equations of motion in vector form, we write

$$\mathbf{V}^{\cdot} = \frac{c\beta}{m}\mathbf{e} + \mathbf{g}, \qquad \mathbf{e}\cdot\mathbf{e} = 1, \qquad m^{\cdot} = -\beta \tag{2.1}$$

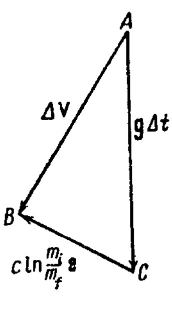

Fig. 1

We shall solve the problem of finding max t_f for a prescribed m_f . Using L.S.Pontriagin's method, we find

$$H = \boldsymbol{\lambda}\left(\frac{c\beta}{m}\mathbf{e} + \mathbf{g}\right) - \lambda_m\beta$$

$$\boldsymbol{\lambda}^{\cdot} = 0, \qquad \mathbf{e} = \frac{\boldsymbol{\lambda}}{|\boldsymbol{\lambda}|} = \text{const} \tag{2.2}$$

Integrating (2.1) and noting that $\mathbf{e}$ is constant, we obtain

$$\Delta\mathbf{V} = c\ln\frac{m_i}{m_f}\mathbf{e} + \mathbf{g}\Delta t$$

From the triangle ABC (Fig.1) we determine the value of Δt. Evidently, in this case $\mathbf{e}\cdot\mathbf{g}\leqslant 0$. The condition

$$\int_{t_i}^{t_f} \beta\, dt = m_i - m_f \qquad (2.3)$$

is imposed for β and not Gorelov's condition where β/m = const.

In the general case, condition (2.3) is satisfied by infinitely many solutions.

If we wish to solve the problem of finding $\max m_f$ for an unspecified t_f, then the side BC (Fig.1) must be a minimum. Let $\Delta\mathbf{V}$ have a vertical component directed downward and let β_{max} be sufficiently large. Then the triangle ABC will be a right triangle and the quantities Δt and m_f will be uniquely defined. As before, β will satisfy the condition (2.3).

An example of such non-uniqueness of the solution was indicated by Leitmann [4] (m_f is prescribed and the horizontal component of velocity is made a maximum).

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